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The Dispersion Relations For MHD Waves in Cylindrical Flux Tubes

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1. INTRODUCTION

We start with the ideal MHD equations, assuming small perturbations for each physical parameter, and combining the equations to get a differential equation containing the parameter to be solved. Then by utilizing the boundary conditions inside and outside of the cylindrical flux tube, the solution to the differential equation can be obtained.

In the following sections, we will treat the problem in cylindrical coordinate (the three base vectors are $\mathbf{e_r}$, $\mathbf{e}_\theta$ and $\mathbf{e_z}$). We assume the radius of the cylinder as $r = a$, and the two typical speeds used during the derivation are *local sound speed* $c_{s,i}$ and *local Alfvén speed* $v_{A,i}$. The index $i = o,\ e$ represents parameters in the interior and exterior regions, respectively.

2. BASIC MHD EQUATIONS

The initial equations are ideal MHD equations (ignoring collisions and viscosity):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \nabla p + \rho g + \frac{1}{\mu_0} [(\nabla \times \mathbf{B}) \times \mathbf{B}] = 0 \tag{2}$$

$$\nabla \times (\mathbf{v} \times \mathbf{B}) - \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{3}$$

3. THE INTRODUCTION OF SMALL PERTURBATIONS

Assume a small perturbation in the parameters:

$\rho = \rho_0 + \rho_1, \quad p = p_0 + p_1, \quad \mathbf{v} = \mathbf{v_1}, \quad \mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1,$

We treat the perturbations as local phenomena, and neglect the large-scale gradients of macroscopic parameters: $\rho g = 0$, $\nabla \rho_0 = 0$, $\nabla p_0 = 0$, and $\nabla \times B_0 = 0$. The magnetic field $\mathbf{B}$ only has z-component: $\mathbf{B} = (0,\ 0,\ B)$.

By introducing these small perturbations, the aforementioned equations (1)-(3) are written as:

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}_1 = 0 \tag{4}$$

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + \nabla[p_1 + \frac{1}{\mu_0} \nabla(\mathbf{B}_0 \cdot \mathbf{B}_1)] - \frac{1}{\mu_0}(\mathbf{B}_0 \cdot \nabla)\mathbf{B}_1 = 0 \tag{5}$$

$$\frac{\partial \mathbf{B}_1}{\partial t} + \mathbf{B}_0(\nabla \cdot \mathbf{v}_1) - (\mathbf{B}_0 \cdot \nabla)\mathbf{v}_1 = 0 \tag{6}$$

In Eq. (5), we can write $p_T = p_1 + \frac{1}{\mu_0}\nabla(\mathbf{B}_0 \cdot \mathbf{B}_1)$ as the total pressure perturbation (gas pressure plus magnetic pressure perturbations).

The above procedure converts Eq. (2) to Eq. (5) which is expressed by the total pressure perturbation p_T, meanwhile, we can also change Eq. (2) to make it contain the density perturbation ρ_1:

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} + c_s^2 \nabla \rho_1 + \frac{1}{\mu_0}[\nabla(\mathbf{B}_0 \cdot \mathbf{B}_1) - (\mathbf{B}_0 \cdot \nabla)\mathbf{B}_1] = 0 \tag{7}$$

4. FURTHER DERIVATIONS

$\frac{\partial}{\partial t}(5)$ gives

$$\rho_0 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} + \nabla \cdot (\frac{\partial p_T}{\partial t}) - \frac{1}{\mu_0}(\mathbf{B}_0 \cdot \frac{\partial^2 \mathbf{B}_1}{\partial z \partial t}) = 0 \tag{8}$$

Combine Eq. (6) and Eq. (8) and know that local Alfvén speed $v_A = \frac{B}{\sqrt{\mu_0 \rho}}$:

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} + \frac{\nabla}{\rho_0} \cdot \frac{\partial p_T}{\partial t} - v_A^2(\frac{\partial^2 \mathbf{v}_1}{\partial z^2} - \mathbf{e}_z \nabla \cdot \mathbf{v}_1) = 0 \tag{9}$$

Take the divergence of the z-component of Eq. (8) (the last two terms in Eq. (8) will cancel):

$$\frac{\partial^2}{\partial t^2}(\nabla \cdot \mathbf{v}_1) + \frac{1}{\rho_0}\nabla^2(\frac{\partial p_T}{\partial t}) = 0 \tag{10}$$

From the r-component of Eq. (9) we get:

$$\rho_0 \frac{\partial^2 v_r}{\partial t^2} - \rho_0 v_A^2 \frac{\partial^2 v_r}{\partial z^2} + \frac{\partial^2}{\partial r^2}\frac{\partial p_T}{\partial t} = 0 \tag{11}$$

$\frac{\partial}{\partial t}$(7) gives

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} = c_s^2 \nabla(\nabla \cdot \mathbf{v}_1) + v_A^2[-\nabla(\frac{\partial v_z}{\partial z} - \nabla \mathbf{v}_1) + \frac{\partial}{\partial z}(\frac{\partial \mathbf{v}_1}{\partial z} - \mathbf{e}_z \nabla \cdot \mathbf{v}_1)] \tag{12}$$

Set $\Delta = \nabla \cdot \mathbf{v}_1$ and $\Gamma = \frac{\partial v_z}{\partial z}$, the z-component and divergence of Eq. (7) give:

$$\frac{\partial^2 v_z}{\partial t^2} = c_s^2 \frac{\partial \Delta}{\partial z} \tag{13}$$

$$\frac{\partial^2 \Delta}{\partial t^2} = (c_s^2 + v_A^2)\nabla^2 \Delta - v_A^2 \nabla^2 \Gamma = 0 \tag{14}$$

Suppose both p_T and Δ have the form of $R(r)exp(i\omega t + in\theta + ikz)$:

$$p_T = -\frac{i\rho_0 \omega}{k^2}\Delta + C \tag{15}$$

C is a constant.

Eqs. (13) and (14) can be combined to be:

$$\frac{\partial^4 \Delta}{\partial t^4} - (c_s^2 + v_A^2)\frac{\partial^2}{\partial t^2}\nabla^2 \Delta + c_s^2 v_A^2 = 0 \tag{16}$$

Take the form $\Delta = R(r)exp(i\omega t + in\theta + ikz)$ into Eq. (16):

$$\frac{d^2 R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} - [\frac{n^2}{r^2} + k^2 - \frac{\omega^4}{(c_s^2 + v_A^2)\omega^2 - k^2 c_s^2 v_A^2}]R(r) = 0 \tag{17}$$

Set $m^2 = k^2 - \frac{\omega^4}{(c_s^2 + v_A^2)\omega^2 - k^2 c_s^2 v_A^2} = \frac{(k^2 c_s^2 - \omega^2)(k^2 v_A^2 - \omega^2)}{(c_s^2 + v_A^2)(k^2 c_T^2 - \omega^2)}$,

and $c_T = \frac{c_s v_A}{\sqrt{c_s^2 + v_A^2}}$ is the so-called *tube speed.*

5. DISPERSION RELATION OF COMPRESSIBLE WAVES

Eq. (17) is a modified Bessel equation, and its solution is modified Bessel functions.

5.1. *Bessel Equation*

Bessel equation is expressed in Cartesian coordinate as $\frac{d^2y(x)}{dx^2} + \frac{1}{x}\frac{dy(x)}{dx} + (1 - \frac{\nu^2}{x^2})y(x) = 0$, by substituting $x = mr,\ y(x) = R(r)$, we have Bessel equation in cylindrical coordinate: $\frac{1}{r}\frac{d}{dr}(r\frac{dR(r)}{dr}) + (m^2 - \frac{n^2}{r^2})R(r) = 0$.

5.2. *Modified Bessel Equation*

If we let $x = ix$ in Bessel equation, we than get the modified Bessel equation: $\frac{d^2y(x)}{dx^2} + \frac{1}{x}\frac{dy(x)}{dx} - (1 + \frac{\nu^2}{x^2})y(x) = 0$.

The solution of modified Bessel equation is $y = AI_\nu(x) + BK_\nu(x)$. $I_\nu(x)$ is the modified Bessel function of the first kind; when $\nu > 0$, $I_\nu(0) = 0$. $K_\nu(x)$ is the modified Bessel function of the second kind; $K_\nu(0) \to \infty$.

5.3. *Solution of the Differential Equation*

The solution of Eq. (17) is $R(r) = AI_n(mr) + BK_n(mr)$ (the radius of the cylinder is a).

Inside the cylinder, because when $r = 0$, $R(r)$ must be finite, $B = 0 \Rightarrow R(r) = AI_n(mr)$ $(r \leq a)$; outside the cylinder, $I_n(mr)$ is exponentially decreasing, so we take $R(r) = BK_n(mr)$ $(r > a)$.

Now we review Eqs. (11) and (15):

$$\rho_0\frac{\partial^2 v_r}{\partial t^2} - \rho_0 v_A^2\frac{\partial^2 v_r}{\partial z^2} + \frac{\partial}{\partial r}\frac{\partial p_T}{\partial t} = 0 \quad (11)$$

$$p_T = -\frac{i\rho_0\omega}{k^2}\Delta + C \quad (15)$$

If we rewrite v_r in the form of $v(r)exp(i\omega t + in\theta + ikz)$, then from Eq. (10) we have:

$$v_r = \frac{m\omega^2}{k^2(\omega^2 - k^2v_A^2)}\frac{dR(r)}{dr} \quad (18)$$

Now we already have the solution of $\Delta = \nabla \cdot \mathbf{v}_1$, and we know the total pressure perturbation p_T and radial velocity perturbation v_r are both related with Δ (Eqs. (15) and (18), respectively).

The boundary conditions are: radial velocity perturbation v_r and total pressure p_T are continuous. Hence, we have

$$v_{r,o}(r=a-) = v_{r,e}(r=a+) \Rightarrow \frac{m_o I_n'(m_o a)}{\omega^2 - k^2 v_{A,o}^2} = \frac{m_e K_n'(m_e a)}{\omega^2 - k^2 v_{A,e}^2} \tag{19}$$

$$p_{T,o}(r=a-) = p_{T,e}(r=a+) \Rightarrow \rho_o I_n(m_o a) = \rho_e K_n(m_e a) \tag{20}$$

Eq. (19)/ Eq. (20) gives the **dispersion relation** for compressible MHD waves:

$$\rho_e(\omega^2 - k^2 v_{A,e}^2) m_o \frac{I_n'(m_o a)}{I_n(m_o a)} = \rho_o(\omega^2 - k^2 v_{A,o}^2) m_e \frac{K_n'(m_e a)}{K_n(m_e a)} \tag{21}$$

Note here $m_i^2 = \dfrac{(k^2 c_s^2 - \omega^2)(k^2 v_A^2 - \omega^2)}{(c_s^2 + v_A^2)(k^2 c_T^2 - \omega^2)}$.

n is an integer, the azimuthal modal structures of waves are determined by n in the dispersion relation.

- $n = 0$: *sausage* modes
- $n = 1$: *kink* modes
- $n > 1$(n is integer): *flute* or *balloning* modes

6. DISPERSION RELATION FOR INCOMPRESSIBLE SITUATIONS

The above derivations are based on the fact that the plasma is compressible, which means we have $p_T \neq 0$. If $p_T = 0$, meaning that the plasma is incompressible (the compressibility term $\nabla \cdot \mathbf{v}_1 = 0$), then Eq. (8) is reduced to

$$\rho_0 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} - \frac{1}{\mu_0}(\mathbf{B}_0 \cdot \frac{\partial^2 \mathbf{B}_1}{\partial z \partial t}) = 0 \tag{22}$$

And Eq. (6) is reduced to

$$\frac{\partial \mathbf{B}_1}{\partial t} - (\mathbf{B}_0 \cdot \nabla)\mathbf{v}_1 = \frac{\partial \mathbf{B}_1}{\partial t} - \mathbf{B}_0 \cdot \frac{\partial \mathbf{v}_1}{\partial z} = 0 \tag{23}$$

Substitute Eq. (23) into Eq. (22):

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \frac{B_0^2}{\mu_0 \rho_0} \frac{\partial^2 \mathbf{v}_1}{\partial z^2} \tag{24}$$

Take the form $\mathbf{v}_1 = v_1(r)exp(i\omega t + in\theta + ikz)$ into Eq. (24):

$$\omega^2 = v_A^2 k^2 \tag{25}$$

Eq. (25) describes a pure Alfvén wave propagating along the field lines, which is the torsional Alfvén wave in flux tubes.

7. CONCLUSION

In conclusion, we have two forms of dispersion relation:

- For incompressible situation:

$$\omega^2 = v_A^2 k^2$$

- For compressible situations:

$$\rho_e(\omega^2 - k^2 v_{A,e}^2) m_o \frac{I_n'(m_o a)}{I_n(m_o a)} = \rho_o(\omega^2 - k^2 v_{A,o}^2) m_e \frac{K_n'(m_e a)}{K_n(m_e a)}$$

($I_n'(m_o r)$ and $K_n'(m_e r)$ are the derivatives of I_n and K_n, i.e., $I_n' = \dfrac{dI_n(m_o r)}{dr}$)

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